

ON THE LAPLACE TRANSFORM OF SOME QUADRATIC FORMS AND THE EXACT DISTRIBUTION OF THE SAMPLE VARIANCE FROM A GAMMA OR UNIFORM PARENT DISTRIBUTION

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Abstract. From a suitable integral representation of the Laplace transform of a positive semi-definite quadratic form of independent real random variables with not necessarily identical densities a univariate integral representation is derived for the cumulative distribution function of the sample variance of i.i.d. random variables with a gamma density, supplementing former formulas of the author. Furthermore, from the above Laplace transform Fourier series are obtained for the density and the distribution function of the sample variance of i.i.d. random variables with a uniform distribution. This distribution can be applied e.g. to a statistical test for a scale parameter.

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1. Introduction

Exact results for the distribution of the sample variance s^2 from non-normal samples are hardly found. Early investigations, concerning s^2 from a sample of gamma random variables can be traced at least to Craig (1929) and Pearson (1929), who used moment approximations, later investigated more thoroughly by Bowman and Shenton (1983). For general distributions various approximations were proposed by Box (1953), Roy and Tiku (1962), Tan and Wong (1977) and by Mudholkar and Trivedi (1981). The latter authors recommend transformations of the Wilson-Hilferty type. For a mixture of two normal distributions see also chapter 4.7 in Mathai and Provost (1992). This monograph treats quadratic forms, mainly from normal random variables. For the cdf of a positive semi-definite quadratic form of normal r.v. a simple integral representation on $[0, \pi]$ is given by formula (31) in Royen (2007a) as the univariate special case of some multivariate distributions. The interest in the distribution of s^2 or $1/s$ was also stimulated by the estimation of process capability indices, see e.g. Pearn, Kotz and Johnson (1992) and Pearn and Kotz (2006).

In this paper the cumulative distribution function (cdf) of $(n-1)s^2 = \sum_{j=1}^n (X_j - \bar{X})^2$ is derived for i.i.d. random variables X_j with a gamma density $x^{\alpha-1}e^{-x}/\Gamma(\alpha)$ and for uniformly distributed X_j . The first distribution was already given by a double series

in Royen (2007b), but now a univariate integral representation of the cdf is derived in section 3 avoiding the recursive computation of the coefficients of the above series. The cited paper contains also the exact distribution of $\left(\sum_{j=1}^n X_j^2\right)^{1/2} / \sum_{j=1}^n X_j$ including Greenwood's statistic if $\alpha = 1$. The distribution of $(n-1)s^2$ from a sample of uniformly distributed random variables is obtained by a Fourier series in section 4 and applied to a statistical test for a scale parameter.

Both these distributions are derived from a suitable integral representation of the Laplace transform (LT) of an " m -factorial" positive semi-definite quadratic form

$$Q = X'(D - CC')X \quad (1)$$

of independent real random variables $X' = (X_1, \dots, X_n)$ with not necessarily identical densities, with positive definite $D = \text{diag}(d_1, \dots, d_n)$ and an $n \times m$ -matrix $C = (c_{j\mu})$ of rank m with real or pure imaginary columns c_μ .

In particular, with i.i.d. random variables and $m = 1$ the form Q is reduced to $(n-1)s^2$ if D is the identity matrix and $c_j = 1/\sqrt{n}$ for all j . The LT of Q is given in the following section.

Formulas from the handbook of mathematical functions by Abramowitz and Stegun are cited by A.S. and their number. The standard normal density is denoted by φ and \mathcal{E} means "expectation".

2. The Laplace transform of an m -factorial positive semi-definite quadratic form

The LT or Fourier transform (FT) of a sum of independent random variables can be written as a product of univariate integral transforms. To preserve this advantage for a quadratic form of the type in (1) as far as possible its LT is represented by an m -variate mixture over a product of univariate integral transforms. This is accomplished essentially by the equation

$$\int_{-\infty}^{\infty} \exp(yz) \varphi(y) dy = \exp(z^2/2), z \in \mathbb{C}. \quad (2)$$

Inspite of its simplicity the result is stated formally as a theorem.

THEOREM 1. *Let $Q = X'(D - CC')X$ be a positive semidefinite quadratic form of the type described in (1). Then the LT of Q is given by*

$$\mathcal{E}(\exp(-tQ)) = \int_{\mathbb{R}^m} \left(\prod_{j=1}^n \mathcal{E}(\exp(-td_j X_j^2 \pm \sqrt{2t} X_j \sum_{\mu=1}^m c_{j\mu} y_\mu)) \right) \cdot \prod_{\mu=1}^m \varphi(y_\mu) dy_\mu \quad (3)$$

with the standard normal density φ , $t \geq 0$, all $d_j > 0$.

REMARK: Replacing t by $-it$ in (3) the characteristic function of Q is obtained if Q is definite or indefinite and absolutely bounded.

With i.i.d. random variables, $m = 1$, $d_j \equiv 1$, $c_j \equiv 1/\sqrt{n}$ we get

$$\begin{aligned}\mathcal{E}(\exp(-tQ)) &= \mathcal{E}(\exp(-t(n-1)s^2)) \\ &= \int_{-\infty}^{\infty} (\mathcal{E}(\exp(-tX^2 \pm \sqrt{2t/n}Xy)))^n \varphi(y) dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mathcal{E}(\exp(-(\sqrt{t}X \mp y/\sqrt{2n})^2)))^n dy.\end{aligned}\quad (4)$$

PROOF OF THEOREM 1 : Let be $m = 1$:

$$\begin{aligned}\mathcal{E}(\exp(-tQ)) &= \mathcal{E}\left(\exp\left(-t \sum_{j=1}^n d_j X_j^2 + t\left(\sum_{j=1}^n c_j X_j\right)^2\right)\right) \\ &= \mathcal{E}\left(\exp\left(-t \sum_{j=1}^n d_j X_j^2\right) \int_{-\infty}^{\infty} \exp\left(\pm y \sqrt{2t} \sum_{j=1}^n c_j X_j\right) \varphi(y) dy\right),\end{aligned}$$

applying (2) with $z = \pm \sqrt{2t} \sum_j c_j X_j$.

For real c_j the integrand is only positive. Thus, equation (3) is obtained by changing the order of integration. For imaginary values c_j the integrand is majorized by $\exp(-tX'DX)\varphi(y)$ and equation (3) follows in the same way. The generalization to $m > 1$ is obvious and needs no further considerations.

3. The exact distribution of the sample variance from a gamma parent distribution

We need a moment generating function M_α and a "kernel" K_β :

$$\begin{aligned}M_\alpha(z) &:= (\Gamma(\alpha))^{-1} \int_0^\infty \exp(-x^2) x^{\alpha-1} \exp(xz) dx \\ &= (2\Gamma(\alpha))^{-1} \sum_{k=0}^{\infty} \Gamma((\alpha+k)/2) z^k / k! \\ &= (2\Gamma(\alpha))^{-1} \left(\Gamma\left(\frac{\alpha}{2}\right) {}_1F_1\left(\frac{\alpha}{2}, \frac{1}{2}, \frac{z^2}{4}\right) + \Gamma\left(\frac{\alpha+1}{2}\right) z {}_1F_1\left(\frac{\alpha+1}{2}, \frac{3}{2}, \frac{z^2}{4}\right) \right), \\ &\quad \alpha > 0, z \in \mathbb{C}.\end{aligned}\quad (5)$$

If $\alpha \in \mathbb{N}$ then

$$M_\alpha(z) = (2\Gamma(\alpha))^{-1} \sqrt{\pi} \left(\frac{d}{dz}\right)^{\alpha-1} (\exp(z^2/4) \operatorname{erfc}(-z/2)). \quad (6)$$

$$\begin{aligned}K_\beta(r, z) &:= \sum_{k=0}^{\infty} (\Gamma(\beta + k/2))^{-1} H e_k(z) (-r)^k / k! \\ &= \sum_{k=0}^{\infty} (\Gamma(\beta + k/2))^{-1} {}_0F_1(\beta + k/2, -r^2/2) (-rz)^k / k!, \\ &\quad \beta > 0, r \geq 0, z \in \mathbb{C},\end{aligned}\quad (7)$$

where $He_k(z) = (-1)^k \varphi^{(k)}(z)/\varphi(z) = 2^{-k/2} H_k(z/\sqrt{2})$ with the Hermite polynomials H_k .

The equivalence of the two series in (7) follows by inversion of the LT

$$\begin{aligned} t^{-\beta} \varphi(z + t^{-1/2})/\varphi(z) &= t^{-\beta} \exp(-z/\sqrt{t}) \exp(-1/(2t)) \\ &= \exp(-1/(2t)) \sum_{k=0}^{\infty} t^{-\beta-k/2} (-z)^k / k! = \sum_{k=0}^{\infty} (-1)^k He_k(z) t^{-\beta-k/2} / k! \end{aligned} \quad (8)$$

with $t > 0$ and any fixed z . The inversion provides the two corresponding series for $x^{\beta-1} K_{\beta}(\sqrt{x}, z)$.

From (7) we obtain also an integral representation of K_{β} by well known functions. Let be

$$\begin{aligned} w_{\beta}(y) &= \sum_{k=0}^{\infty} (\Gamma(\beta + k/2))^{-1} y^k \\ &= (\Gamma(\beta))^{-1} {}_1F_1(1, \beta, y^2) + (\Gamma(\beta + 1/2))^{-1} y {}_1F_1(1, \beta + 1/2, y^2), \quad y \in \mathbb{C}. \end{aligned}$$

After Kummer's transformation (A.S.13.1.27) we find with (A.S.13.6.10) and $G_{\beta}(z) = (\Gamma(\beta))^{-1} \int_0^z x^{\beta-1} e^{-x} dx$

$$w_{\beta}(y) = \exp(y^2) y^{-2(\beta-1)} (G_{\beta-1}(y^2) + G_{\beta-1/2}(y^2)), \quad \beta > 1.$$

In particular with $\beta = 1 + m \in \mathbb{N}$ we have

$$\begin{aligned} w_{\beta}(y) &= y^{-2m} \left(\exp(y^2) \operatorname{erfc}(-y) - \sum_{j=0}^{2m-1} y^j / \Gamma(1 + j/2) \right), \\ w_{1/2}(y) &= y \exp(y^2) \operatorname{erfc}(-y) + \pi^{-1/2}, \\ w_{1/2+m}(y) &= y^{-1} (w_m(y) - 1/(m-1)!). \end{aligned}$$

Thus, with any $\rho > 0$, we obtain

$$K_{\beta}(r, z) \varphi(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(z + \rho e^{i\psi}) w_{\beta}(\rho^{-1} r e^{-i\psi}) d\psi.$$

K_{β} will be used only with real z . In this case the real part of the integrand is integrated on $[0, \pi]$.

Furthermore, from the 2^{nd} series in (7) and the integral representation of the Bessel functions $J_{\beta-1+k/2}$ it follows

$$\begin{aligned} K_{\beta}(r, z) &= (2/\sqrt{\pi}) \int_0^{\pi/2} \left({}_0F_2\left(\frac{1}{2}, \beta - \frac{1}{2}, \frac{1}{4} r^2 z^2 \sin^2 \vartheta\right) / \Gamma(\beta - \frac{1}{2}) - \right. \\ &\quad \left. (r z \sin \vartheta) {}_0F_2\left(\frac{3}{2}, \beta, \frac{1}{4} r^2 z^2 \sin^2 \vartheta\right) / \Gamma(\beta) \right) (\sin^2 \vartheta)^{\beta-1} \cos(r\sqrt{2} \cos \vartheta) d\vartheta, \quad \beta > \frac{1}{2}. \end{aligned}$$

For the proof of the following theorem 2 we need some knowledge concerning the behaviour of the functions M_{α} and K_{β} for large $|z|$, z real. From the integral in (5) and the Tauber theorem for Laplace transforms of measures (e.g. Feller, chapter 13) it follows

$$\lim_{z \rightarrow -\infty} |z|^{\alpha} M_{\alpha}(z) = 1. \quad (9)$$

With (A.S.9.1.62) and (A.S.9.1.69) we have $|{}_0F_1(\beta + k/2, -r^2/2)| \leq 1$. Then, from $\lim_{k \rightarrow \infty} \varepsilon^{-k}(\Gamma(\beta + k/2))^{-1} = 0$ for all $\varepsilon > 0$, we obtain with the second series in (7)

$$K_\beta(r, z) = o(e^{\varepsilon r|z|}), \quad \varepsilon > 0, \quad z \rightarrow -\infty. \quad (10)$$

With (A.S.22.14.17) we get $|He_k(z)|/k! < 1.1e^{z^2/4}/\sqrt{k!}$ and consequently

$$|K_\beta(r, z)| < 1.1e^{z^2/4} \sum_{k=0}^{\infty} (\Gamma(\beta + k/2)\sqrt{k!})^{-1} r^k, \quad z \in \mathbb{R}. \quad (11)$$

From the last representation in (5) and (A.S.13.1.14) it follows the asymptotic relation

$$M_\alpha(z) \simeq (\sqrt{\pi}/\Gamma(\alpha)) \exp(z^2/4)(z/2)^{\alpha-1}, \quad z \rightarrow \infty. \quad (12)$$

Finally, replacing t by $-it$ in (8) we find with $z > 0$ and $\beta > 1$ by the Fourier inversion formula

$$\begin{aligned} |x^{\beta-1} K_\beta(\sqrt{x}, z)| &\leq \pi^{-1} \int_0^\infty t^{-\beta} \exp(-z/\sqrt{2t}) dt \\ &= (2^\beta/\pi) \Gamma(2\beta - 2)/z^{2\beta-2}. \end{aligned} \quad (13)$$

THEOREM 2. *Let X_1, \dots, X_n be i.i.d. random variables with a gamma density $x^{\alpha-1} \exp(-x)/\Gamma(\alpha)$. With the functions M_α from (5) and K_β from (7) the cdf of $Q = \sum_{i=1}^n (X_i - \bar{X})^2$, $n \geq 2$, is given by*

$$\Pr\{Q \leq x\} = x^{\alpha n/2} \int_{-\infty}^{\infty} K_{\alpha n/2+1}(\sqrt{nx/2}, z) (M_\alpha(\sqrt{2/n} z))^n \varphi(z) dz. \quad (14)$$

PROOF. Due to the relations (9)...(13) with $\beta = \alpha n/2 + 1$ the integral in (14) is absolutely convergent for all $\alpha > 0$ and $x \geq 0$. In particular, the integrand is at least $O(z^{-n})$ for $z \rightarrow \infty$. Furthermore, the LT

$$\int_0^\infty e^{-tx} x^{\alpha n/2} \left(\int_{-\infty}^{\infty} |K_{\alpha n/2+1}(\sqrt{nx/2}, z)| (M_\alpha(\sqrt{2/n} z))^n \varphi(z) dz \right) dx$$

exists for all $t > 0$. Thus, changing the order of integration, the LT of

$$x^{\alpha n/2} \int_{-\infty}^{\infty} K_{\alpha n/2+1}(\sqrt{nx/2}, z) (M_\alpha(\sqrt{2/n} z))^n \varphi(z) dz$$

is given by

$$t^{-(\alpha n/2+1)} \int_{-\infty}^{\infty} \left(\varphi(z + \sqrt{n/(2t)})/\varphi(z) \right) \left(M_\alpha(\sqrt{2/n} z) \right)^n \varphi(z) dz. \quad (15)$$

On the other hand, applying equation (4) to a sample of i.i.d. random variables with density $x^{\alpha-1} \exp(-x)/\Gamma(\alpha)$, we find the LT of the density of Q .

$$\begin{aligned}\mathcal{E}(e^{-tQ}) &= (\Gamma(\alpha))^{-n} \int_{-\infty}^{\infty} \left(\int_0^{\infty} \exp(-tx^2 + \sqrt{2t/n} xy - x) x^{\alpha-1} dx \right)^n \varphi(y) dy \\ &= (\Gamma(\alpha))^{-n} t^{-\alpha n/2} \int_{-\infty}^{\infty} \left(\int_0^{\infty} \exp(-x^2 + (\sqrt{2/n} y - 1/\sqrt{t})x) x^{\alpha-1} dx \right)^n \varphi(y) dy \\ &= (\Gamma(\alpha))^{-n} t^{-\alpha n/2} \int_{-\infty}^{\infty} \left(\int_0^{\infty} \exp(-x^2 + \sqrt{2/n} xz) x^{\alpha-1} dx \right)^n \varphi(z + \sqrt{n/(2t)}) dz\end{aligned}$$

with the substitution $z = y - \sqrt{n/(2t)}$. Therefore $t^{-1}\mathcal{E}(e^{-tQ})$ coincides with (15) which completes the proof.

In the same way the more general formula

$$\Pr\{Q \leq x\} = x^{\alpha/2} \int_{-\infty}^{\infty} K_{\alpha/2+1}(\sqrt{nx/2}, z) \left(\prod_{j=1}^n M_{\alpha_j}(\sqrt{2/n} z) \right) \varphi(z) dz, \quad \alpha = \sum_{j=1}^n \alpha_j$$

is proved for independent $\Gamma(\alpha_j)$ -distributed random variables X_j .

4. The exact distribution of the sample variance from a uniform parent distribution

Let X_1, \dots, X_n be i.i.d. random variables, uniformly distributed on the interval $[0, 1]$. The maximum Q_{\max} of $Q = \sum_{i=1}^n (X_i - \bar{X})^2$ is given by $n/4$ if n is even and by $(n^2 - 1)/(4n)$ otherwise. These values are reached if $X_i(1 - X_i) = 0$, $i = 1, \dots, n$, with exactly $[n/2]$ solutions $X_i = 1$ or $[n/2]$ solutions $X_i = 0$. For a formal proof set $Y_i = 2X_i - 1 = \sin \Phi_i$. The coordinates of the stationary points of $4Q = \sum_{i=1}^n \sin^2 \Phi_i - 1/n(\sum_{i=1}^n \sin \Phi_i)^2$ satisfy $Y_i = \pm 1$ or $Y_i = \bar{Y}$. It is easy to show that the maximum is only attained if $Y_i = \bar{Y}$ does not occur. Then the result follows by maximizing $Q = k(n - k)/n$, $k = 1, \dots, n - 1$.

From (4) the LT of Q follows by straightforward calculation. Hence, the Fourier transform \hat{f} of the density f of Q is given by

$$\hat{f}(t) = \sqrt{n}(\sqrt{\pi}/2)^{n-1} (i/t)^{n/2} \int_0^{\infty} \left(\operatorname{erf}\left(y + \frac{1}{2}\sqrt{-it}\right) - \operatorname{erf}\left(y - \frac{1}{2}\sqrt{-it}\right) \right)^n dy \quad (16)$$

and the Fourier sinus coefficients b_k of $f(x)$, $0 \leq x \leq Q_{\max}$, are

$$\begin{aligned}b_k &= (2/Q_{\max}) \Im m(\hat{f}(t_k)), \\ t_k &= \begin{cases} 4k\pi/n & , \quad n \text{ even} \\ 4k\pi n/(n^2 - 1) & , \quad n \text{ odd} \end{cases}.\end{aligned} \quad (17)$$

However, for numerical reasons, any values of $\operatorname{erf}(z)$ with $|\Im m(z)| > |\Re e(z)|$ should be avoided within the integral $\hat{f}(t)$. Therefore, the path of integration is shifted by $1/2\sqrt{-it}$. With Cauchy's integral theorem for holomorphic functions and

$$\lim_{y \rightarrow \infty} \operatorname{erf}(y \pm \sqrt{-it}) - \operatorname{erf}(y) = 0,$$

uniformly on intervals $0 \leq t \leq T$, the integral

$$\int_0^\infty \left(\operatorname{erf}\left(y + \frac{1}{2}\sqrt{-it}\right) - \operatorname{erf}\left(y - \frac{1}{2}\sqrt{-it}\right) \right)^n dy$$

in (16) can be replaced by the identical value

$$\begin{aligned} & \int_0^\infty \left(\operatorname{erf}(y + \sqrt{-it}) - \operatorname{erf}(y) \right)^n dy \\ & + \frac{1}{2}\sqrt{-it} \int_0^1 \left(\operatorname{erf}\left(\frac{1}{2}\sqrt{-it}(u+1)\right) - \operatorname{erf}\left(\frac{1}{2}\sqrt{-it}(u-1)\right) \right)^n du. \end{aligned} \quad (18)$$

With the values t_k from (17) this leads to

THEOREM 3. *Let X_1, \dots, X_n be independent random variables, uniformly distributed on the interval $[0, 1]$. The cdf of $Q = \sum_{i=1}^n (X_i - \bar{X})^2$ is given by*

$$\begin{aligned} F(x) &= \Pr\{Q \leq x\} = \frac{1}{2}\sqrt{n}(\sqrt{\pi}/2)^{n-3} \cdot \\ & \left(\int_0^\infty \sum_{k=1}^\infty \Im m\left((i/t_k)^{n/2} \left(\operatorname{erf}(y + \sqrt{-it_k}) - \operatorname{erf}(y) \right)^n\right) (1 - \cos(t_k x))/k dy \right. \\ & \frac{1}{2} \int_0^1 \sum_{k=1}^\infty \Im m\left((i/t_k)^{(n-1)/2} \left(\operatorname{erf}\left(\frac{1}{2}\sqrt{-it_k}(u+1)\right) - \operatorname{erf}\left(\frac{1}{2}\sqrt{-it_k}(u-1)\right) \right)^n\right) \cdot \\ & \left. (1 - \cos(t_k x))/k du \right), \\ 0 \leq x < Q_{\max} &= \begin{cases} n/4 & , \ n \text{ even} \\ (n^2 - 1)/(4n) & , \ n \text{ odd} \end{cases}, \quad n \geq 3. \end{aligned} \quad (19)$$

The density $f(x)$ is obtained by the derivative under the integral at least for $n \geq 4$.

PROOF. The formal series

$$\frac{2}{\pi} \sum_{k=1}^\infty \Im m(\widehat{f}(t_k))(1 - \cos(t_k x))/k \quad (20)$$

for $F(x)$ follows by integration of the formal Fourier sinus series for the density f . Since $F(x)$ is geometrically the intersection volume of the unit n -cube with an n -cylinder of radius $r = \sqrt{x}$ whose axis passes $(0, \dots, 0)$ and $(1, \dots, 1)$, we have $F(x) = O(x^{(n-1)/2})$ and $f(x) = O(x^{(n-3)/2})$ for $x \rightarrow 0_+$. Besides, $f(x)$ tends to zero for $x \rightarrow Q_{\max}$. Hence, f is square integrable for all $n \geq 3$ and consequently the formula (20) provides an absolutely convergent series representation of $F(x)$. After the replacement described in (18) the resulting series differs from (19) only by the change of summation and integration. Since $\operatorname{erf}(z)$ is absolutely bounded on $\{z \in \mathbb{C} \mid |\Im m(z)| \leq |\Re e(z)|\}$ the change is justified for the

2nd integral in (19). The first integral in (19) is divided into $\int_0^1 + \int_1^\infty$ and we get with $y \geq 1$ and $t = t_k > 0$ the estimation

$$\begin{aligned}
& |(\operatorname{erf}(y + \sqrt{-it}) - \operatorname{erf}(y))/\sqrt{-it}| \\
&= \frac{2}{\sqrt{\pi}} |(-it)^{-1/2} \int_y^{y+\sqrt{-it}} \exp(-z^2) dz| \\
&\leq \frac{2}{\sqrt{\pi}} \int_0^1 |\exp(-y^2 - 2uy\sqrt{-it} + itu^2)| du \\
&\leq \frac{2}{\sqrt{\pi}} \int_0^1 \exp(-y^2 - \sqrt{2t} uy) du \\
&= \sqrt{2/\pi} t^{-1/2} (1 - \exp(-\sqrt{2t} y)) \exp(-y^2)/y \\
&< t^{-1/2} \exp(-y^2)/y.
\end{aligned}$$

From this a majorant is obtained for the above integral and theorem 3 is proved.

REMARK. In all the plotted examples the integrands in (19) exhibited a very smooth appearance. Since the integrand of the first integral in (19) tends to zero very rapidly the upper limit of integration can be lowered to moderate values for a numerical integration.

By Mir and Richards (1975) for $r \leq 1/\sqrt{2}$ the elementary formula

$$\Pr\{Q \leq r^2\} = (\sqrt{n} \pi^{(n-1)/2} / \Gamma((n+1)/2)) r^{n-1} - b(n) r^n, \quad n \geq 3 \quad (21)$$

was derived where $b(3) = 2\sqrt{6}$ and $b(n)$ is given by an integral over the $(n-3)$ -unit cube for $n > 3$. The authors have computed $b(n)$ exactly for $n = 3, 4, 5$ and additionally for $n = 6, 7$ by numerical integration. Their values $b(n)$ have been checked by means of (19) and further values $b(n)$ can be computed now by univariate integration, e.g. $b(8) = 13.03951 \dots$, $b(9) = 11.72490 \dots$, $b(10) = 9.90095 \dots$.

The distribution of

$$(Q - \mathcal{E}(Q))(\operatorname{var}(Q))^{-1/2} = (Q - (n-1)/12)((n-1)(2n+3)/(360n))^{-1/2}$$

tends to a standard normal distribution for $n \rightarrow \infty$.

An example for the application of the distribution of Q is the following statistical test. Let Y be continuously distributed with a cdf F and the median m . To test $F = F_0$, where the specified cdf F_0 has the same median, against

$$F_0(y) < F(y), \quad y < m \quad \text{and} \quad F_0(y) > F(y), \quad y > m,$$

we can use the test statistic $Q = \sum_{i=1}^n (U_i - \bar{U})^2$ with $U_i = F_0(Y_i)$, based on a random sample Y_1, \dots, Y_n , and reject $F = F_0$ for large values of Q . In particular, with $F(y) = G((y-m)/\beta)$ and $F_0(y) = G((y-m)/\beta_0)$, $\beta > \beta_0 > 0$, where G denotes a continuous (not normal) cdf with $\operatorname{median}(G) = 0$, this is a test for the scale parameter β . It is supposed that the test works also if the medians of F and F_0 differ only slightly, but the effect to the power is difficult to assess. The simpler test statistic $\sum_{i=1}^n (U_i - 1/2)^2$ with the exact cdf

$$2^{-(n-1)} \pi^{n/2-1} \sum_{k=1}^{\infty} \Im m \left(\left(\operatorname{erf}(\sqrt{-ik\pi/n}) / \sqrt{-ik\pi/n} \right)^n (1 - \cos(4k\pi x/n)) / k, \quad 0 \leq x < n/4 \right)$$

can also be used.

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